

# Dynamic Alignment and Exact Scaling Laws in MHD Turbulence

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The Kolmogorov theory of hydrodynamic turbulence yields an exact relation for the third-order longitudinal velocity structure function, namely  $\langle \delta v_L^3(r) \rangle = -4/5 \epsilon r$ , where  $\delta v_L(r) = [\mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x})] \cdot \mathbf{r}/r$  and  $\epsilon$  is the rate of energy dissipation. One therefore expects the velocity scaling  $\delta v(r) \propto r^{1/3}$ , which leads to the Kolmogorov energy spectrum  $E(k) \propto k^{-5/3}$ . In 1998, Politano and Pouquet found that in magnetohydrodynamic turbulence certain third-order structure functions scale linearly with  $r$ . This, in turn, suggests that the spectrum of MHD turbulence also has the Kolmogorov scaling. However, recent high-resolution direct numerical simulations suggest that the spectrum is  $E(k) \propto k^{-3/2}$ . Here we propose that this apparent contradiction is a manifestation of the phenomenon of scale-dependent dynamic alignment recently discovered in MHD turbulence in [12–14].

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## INTRODUCTION

It is well known that the Kolmogorov theory for isotropic incompressible hydrodynamic turbulence yields the following exact relation for the third order longitudinal structure function of the velocity field in the inertial range (see, e.g., [1]):

$$\langle \delta v_L^3(\mathbf{r}) \rangle = -\frac{4}{5} \epsilon r. \quad (1)$$

Here  $\delta \mathbf{v}(\mathbf{r})$  is the velocity difference between two points separated by the vector  $\mathbf{r}$ ,  $\delta v_L(\mathbf{r}) = \delta \mathbf{v}(\mathbf{r}) \cdot \mathbf{r}/r$  is its longitudinal component and  $\epsilon$  is the rate of energy supply to the system at large scales. In a stationary state, it coincides with the rate of energy cascade toward small dissipative scales, and with the rate of energy dissipation. If one further assumes that the fluctuations are not strong compared to the rms value of  $\delta \mathbf{v}(\mathbf{r})$ , and that  $\delta v(r) \sim \delta v_L(r)$ , one can dimensionally estimate from (1) that  $\langle \delta v^2(\mathbf{r}) \rangle \propto r^{2/3}$ . The Fourier transform of the latter expression then leads to the Kolmogorov spectrum of turbulence,  $E(k) \propto k^{-5/3}$ .

Interestingly, equivalent relations hold for magnetohydrodynamic turbulence. Writing the fluctuating magnetic field and velocity field ( $\mathbf{v}$  and  $\mathbf{b}$ , respectively) in terms of the Elsässer variables  $\mathbf{z} = \mathbf{v} - \mathbf{b}$  and  $\mathbf{w} = \mathbf{v} + \mathbf{b}$ , Politano & Pouquet [2, 3] derived

$$S_{3L}^w(r) \equiv \langle \delta z_L (\delta \mathbf{w})^2 \rangle = -\frac{4}{3} \epsilon^w r, \quad (2)$$

$$S_{3L}^z(r) \equiv \langle \delta w_L (\delta \mathbf{z})^2 \rangle = -\frac{4}{3} \epsilon^z r, \quad (3)$$

where  $\delta z_L$  and  $\delta w_L$  are longitudinal components of  $\delta \mathbf{z}$  and  $\delta \mathbf{w}$ ,  $\epsilon^w$  is the transfer rate of the  $\mathbf{w}$  field and  $\epsilon^z$  is the transfer rate of the  $\mathbf{z}$  field. In a setting with a *nonzero* large-scale field  $\mathbf{B}_0$ , one can derive Eqs. (2, 3)

by assuming that the average is performed over the random forcing and then over all directions of the large-scale field  $\mathbf{B}_0$ . One may expect that the setting with strong nonzero guiding field mimics the inertial range of MHD turbulence, since small-scale eddies are always subject to the strong guiding fields that are locally produced by the long-living large-scale eddies (see, e.g., [7–9]), although such a correspondence principle is, strictly speaking, a conjecture.

If one now follows the analogy with the non-magnetized case and assumes that all typical fluctuations are of the same size ( $\delta z_L \sim \delta w_L \sim \delta z \sim \delta w \sim \delta v \sim \delta b$ ) one derives  $\delta v_r \sim \delta b_r \propto r^{1/3}$ , which leads to the Kolmogorov scaling of the MHD turbulence spectrum. However, this is in disagreement with recent high resolution numerical simulations of strongly magnetized turbulence that reveal the field-perpendicular energy spectrum  $E(k_\perp) \propto k_\perp^{-3/2}$  (see [9–11]). This apparent contradiction of numerical observations with the exact Politano-Pouquet relations (2, 3) motivated our interest in the problem.

In the present paper, we propose that the numerical results are reconciled with the Politano-Pouquet relations if one invokes the phenomenon of scale-dependent dynamic alignment. The essence of the phenomenon is that at each field-perpendicular scale  $r$  ( $\sim 1/k_\perp$ ) in the inertial range, typical shear-Alfvén velocity fluctuations ( $\delta \mathbf{v}_r$ ) and magnetic fluctuations ( $\pm \delta \mathbf{b}_r$ ) tend to align the directions of their polarizations in the field-perpendicular plane. The alignment is stronger for smaller scales, with the alignment angle decreasing with the scale as  $\theta_r \propto r^{1/4}$ . This leads to the scaling of the velocity and magnetic fluctuations,  $\delta v_r \sim \delta b_r \propto r^{1/4}$ , and explains the observed energy spectrum,  $E(k_\perp) \propto k_\perp^{-3/2}$ . This effect was predicted analytically in [12, 13] and verified numerically in [14].

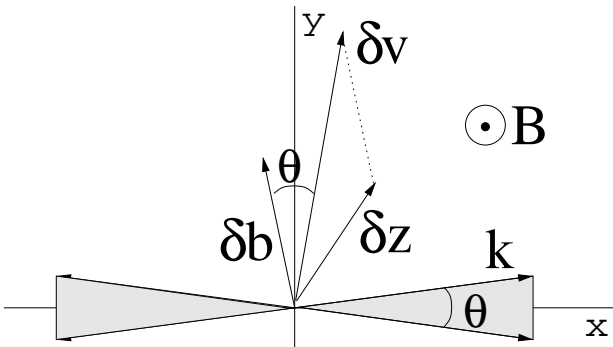


FIG. 1: Sketch of typical velocity and magnetic field fluctuations  $\delta\mathbf{v}_r$  and  $\delta\mathbf{b}_r$  aligned in the field-perpendicular plane within small angle  $\theta_r$ . **\*\*Is  $\theta_0 = \theta$ , or is the shaded cone not what I think it is?\*\*\***

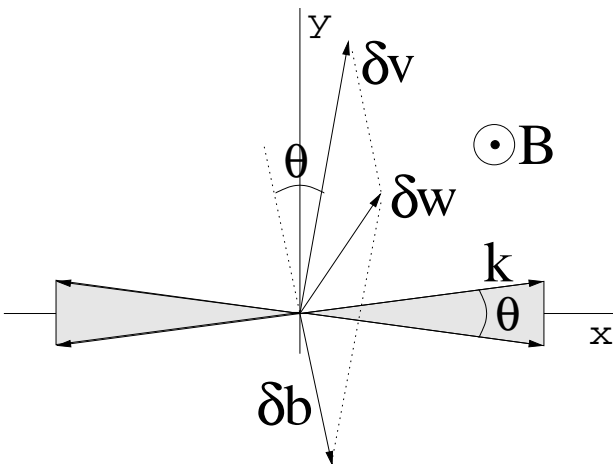


FIG. 2: Sketch of typical velocity and magnetic field fluctuations,  $\delta\mathbf{v}_r$  and  $\delta\mathbf{b}_r$ , aligned in the field-perpendicular plane such that the alignment angle  $\theta_r$  between  $\delta\mathbf{v}_r$  and  $-\delta\mathbf{b}_r$  is small.

### DYNAMIC ALIGNMENT AND POLITANO-POUQUET RELATIONS

There are two possibilities for the dynamic alignment: the velocity fluctuation  $\delta\mathbf{v}_r$  can be aligned either with  $\delta\mathbf{b}_r$  or with  $-\delta\mathbf{b}_r$ . These two cases are presented in Fig. 1 and Fig. 2, respectively. By definition, the amplitudes of  $\delta\mathbf{v}_r$  and  $\delta\mathbf{b}_r$  are of the order of their typical, rms values. Let us determine which configuration provides the dominant contribution to the structure functions (2) and (3). It is easy to see from Fig. 1 and Fig. 2 that when  $\delta z_r \sim \delta w_r$ , both alignment configurations contribute to both structure functions (2) and (3). However, when the magnitudes  $\delta v_r$  and  $\delta b_r$  are very close to each other, i.e. when the corresponding magnitudes  $\delta z_r$  and  $\delta w_r$  are significantly different, the configuration in Fig. 1 provides the dominant contribution to the structure function (2), while that in Fig. 2 dominates the structure function (3).

Without loss of generality, we therefore consider in detail only the structure function  $S_{3L}^w(r)$ , defined in (2), and

concentrate on the contribution provided by the configuration presented in Fig. 1. The directions of the vectors  $\delta\mathbf{v}_r$  and  $\delta\mathbf{b}_r$  are aligned within a small angle  $\theta_r$  in the field-perpendicular plane, in the y-direction, say, and their wave vectors are aligned in the field-perpendicular plane in the x-direction. The dominant contribution to the structure function (2) then comes from the situation in which the point-separation vector  $\mathbf{r}$  lies in the x-direction, since the variation of the fields is strongest in this direction. In this case, the longitudinal projection (i.e., x-component) of  $\delta\mathbf{z}$ ,  $\delta z_L$ , is smaller by a factor  $\theta_r$  than the typical value of  $\delta w_r$ . This introduces an extra factor  $\theta_r$  in the Politano-Pouquet correlation function (2), and one obtains

$$\langle \delta z_L (\delta\mathbf{w})^2 \rangle \sim \theta_r \delta v_r^3. \quad (4)$$

As we demonstrated in [12–14], the scale-dependent dynamic alignment  $\theta_r \propto r^{1/4}$  leads to the scaling of fluctuating fields  $\delta v_r \sim \delta b_r \propto r^{1/4}$ , which explains the numerically observed field-perpendicular energy spectrum,  $E(k_\perp) \propto k_\perp^{-3/2}$ . Quite remarkably, by substituting these scalings into expression (4) we satisfy the scaling relation (2). Thus the earlier mentioned numerical findings are reconciled with the Politano-Pouquet relations if one invokes the phenomenon of scale-dependent dynamic alignment.

In the next section we verify relation (4) numerically. In particular, we measure the following structure functions

$$\tilde{S}_{3L}^w(r) = \langle |\delta z_L| (\delta\mathbf{w})^2 \rangle, \quad (5)$$

$$S_3^w(r) = \langle |\delta\mathbf{w}|^3 \rangle. \quad (6)$$

We use the absolute value of  $\delta z_L$  in calculating (5) to avoid cancellations and slow convergence caused by different signs of  $\delta z_L$ . If our idea expressed by (4) is correct, then  $\tilde{S}_{3L}^w(r) \sim \theta_r \delta v_r^3$  while  $S_3^w(r) \sim \delta v_r^3$  and therefore the ratio of these two functions should give us the alignment angle  $\theta_r$ :

$$\theta_r = \tilde{S}_{3L}^w(r) / S_3^w(r). \quad (7)$$

This angle should scale with the point separation approximately as  $\theta_r \propto r^{1/4}$ .

### NUMERICAL RESULTS

We solve the MHD equations

$$\begin{aligned} \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} &= -\nabla p + (\nabla \times \mathbf{B}) \times \mathbf{B} + \nu \Delta \mathbf{v} + \mathbf{f}, \\ \partial_t \mathbf{B} &= \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \Delta \mathbf{B}, \end{aligned} \quad (8)$$

where  $\mathbf{v}(\mathbf{x}, t)$  is the velocity field,  $\mathbf{B}(\mathbf{x}, t)$  the magnetic field,  $p$  the pressure,  $\mathbf{f}(\mathbf{x}, t)$  the external force, and  $\nu$  and  $\eta$  are the fluid viscosity and resistivity, respectively, using

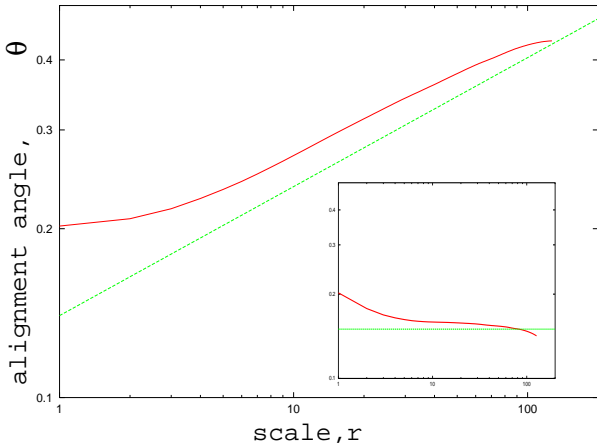


FIG. 3: The relative scaling of structure functions  $\tilde{S}_{3L}^w(r)$  and  $S_3^w(r)$  expressed through the scaling of the alignment angle (7). The angle  $\theta_r$  is plotted vs scale  $r$ . The straight line has the slope 0.23. The insert shows the relative scaling compensated by  $r^{-0.23}$ . **Repeat for  $B_0 = 10$ .**

standard pseudospectral methods. An external magnetic field is applied in  $z$  direction with strength  $B_0 \approx 5$  measured in units of velocity. The periodic domain has a resolution of  $256^3$  mesh points and is elongated in the  $z$  direction, with aspect ratio  $1:1:B_0$ . The external force,  $\mathbf{f}(\mathbf{x}, t)$ , is chosen so as to drive the turbulence at large scales and it satisfies the following requirements: it has no component along  $z$ , it is solenoidal in the  $x-y$  plane, all the Fourier coefficients outside the range  $1 \leq k \leq 2$  are zero, the Fourier coefficients inside that range are Gaussian random numbers with unit variance and amplitude chosen so that the resulting *rms* velocity fluctuations are of order unity, and the individual random values are refreshed independently *on average* every turnover time of the large scale eddies. The Reynolds number is defined as  $Re = U_{rms}L/\nu$ , where  $L$  ( $\sim 1$ ) is the field-perpendicular box size,  $\nu$  is fluid viscosity, and  $U_{rms}$  ( $\sim 1$ ) is the rms value of velocity fluctuations. We restrict ourselves to the case in which magnetic resistivity and fluid viscosity are the same,  $\nu = \eta$ , with  $Re \approx 800$ . The system is evolved until a stationary state is reached (confirmed by observing the time evolution of the total energy of fluctuations). The data are then sampled in intervals of the order of the large-scale eddy turn-over time until the computed correlation functions do not change appreciably. All results presented correspond to averages over these samples (total 16 samples).

To calculate the structure functions (5) and (6), we construct  $\delta\mathbf{z}(r) = \mathbf{z}(\mathbf{x} + \mathbf{r}) - \mathbf{z}(\mathbf{x})$  and  $\delta\mathbf{w}(r) = \mathbf{w}(\mathbf{x} + \mathbf{r}) - \mathbf{w}(\mathbf{x})$ , with  $\mathbf{r}$  in a plane perpendicular to  $\mathbf{B}_0$ . By definition,  $\delta z_L = \delta\mathbf{z}(r) \cdot \mathbf{r}/r$  and  $\delta w_L = \delta\mathbf{w}(r) \cdot \mathbf{r}/r$ . The average is then taken over different positions of the point  $\mathbf{x}$  in that plane, over all such planes in the data cube, and then over all data cubes.

The numerical calculation of expression (7) is shown

in Fig. 3. We find  $\theta_r \propto r^{0.23}$ . The agreement with the analytic prediction  $\theta_r \propto r^{0.25}$  is decent, given that the resolution of our simulations is not large enough to observe a well defined inertial interval, that the magnetic field  $B_0 \approx 5$  is only moderately strong, and that possible small intermittency corrections are not captured by our model. The result presented in (4) and numerically verified in Fig. 3 is the main result of our work.

## DISCUSSION AND CONCLUSION

Dynamic alignment is a known phenomenon of MHD turbulence [15–17]. However, in previous works it essentially meant that decaying MHD turbulence asymptotically reaches the so-called Alfvénic state where either  $\mathbf{v}(\mathbf{x}) \equiv \mathbf{b}(\mathbf{x})$  or  $\mathbf{v}(\mathbf{x}) \equiv -\mathbf{b}(\mathbf{x})$ , depending on initial conditions. It has been realized only recently [12, 13] that the effect is preserved in driven, stationary MHD turbulence, although in quite an interesting fashion. Fluctuations of  $\mathbf{v}$  and  $\pm\mathbf{b}$  tend to align their directions (although their magnitudes are not necessarily equal) and the alignment angle becomes *scale-dependent*, i.e., it decreases with scale as  $\theta_r \propto r^{1/4}$  [12, 13]. The first numerical observations of this phenomenon appeared in [14]. In the present paper we have demonstrated that this phenomenon is consistent with the exact relations known in MHD turbulence due to work by Politano and Pouquet [2, 3]. Our work thus serves as an additional argument in favor of the new model of MHD turbulence developed in [12–14].

Remarkably, both the ideas of the Alfvénic increase of the interaction time, proposed by Iroshnikov and Kraichnan [4, 5], and of critically balanced field-parallel and field-perpendicular cascades, put forward by Goldreich and Sridhar [6], turn out to be consistent with the presented model. It is interesting, however, that the underlying physics of the model is qualitatively different from the physics originally envisaged in either [4, 5] or [6]. In contrast with [4, 5], in our model the turbulence is essentially anisotropic and strong at all scales. In contrast with [6], in our model the turbulent fluctuations are dynamically aligned, their nonlinear interaction is depleted, and the resulting spectrum is different from the prediction of [6].

MHD turbulence plays an essential role in astrophysical phenomena such as the solar wind (e.g. [18]), interstellar scintillation (e.g. [19]), cosmic ray acceleration, propagation, and scattering in the interstellar medium [ADD REFS], and thermal conduction in galaxy **\*\*galactic?\*** clusters (e.g. [20–22]). One of the most important consequences of the scale-dependent dynamic alignment is the energy spectrum of MHD turbulence, which becomes strongly anisotropic with respect to the local magnetic field. The field-perpendicular spectrum takes the form  $E(k_\perp) \propto k_\perp^{-3/2}$ . If one neglects many uncertainties,

the spectrum of turbulence inferred from astrophysical observations is usually consistent with the Kolmogorov spectrum  $k^{-5/3}$ . However, there exist indications in favor of the spectrum  $-3/2$  in scintillation observations [23, 24]. Our theory may provide a natural explanation for such observations.

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