Model flames in the Boussinesq limit: Rising bubbles

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Abstract. Using the Boussinesq buoyancy approximation, we study a bubble of reaction products rising in the reactant fluid under the influence of gravity. Reaction on the surface of the bubble (the flame) results in an increase of the volume of the bubble. We consider fluids with low Prandtl and high Froude numbers (heat diffusion dominates over viscous dissipation, and burning dominates over gravitational effects). We show that, under these conditions, all initially small bubbles follow the same growth pattern, regardless of the flame speed, the reaction type, the gravity, the viscosity, the initial size, and, to some extent, the initial shape of the bubble. In the initial stage of this similarity solution, a bubble grows radially in an essentially motionless fluid until it reaches some critical size, which is determined by the laminar flame speed, the gravitational acceleration, and the Atwood number. Once the bubble reaches the critical size, convection becomes significant and the bubble evolves into a more complicated, mushroom-like shape. The similarity solution is expressed using the critical bubble size for the unit length and the critical size divided by the laminar flame speed as the unit time. (18 August 2006)

1. Introduction

The problem discussed herein arises from attempts to understand the initial stages of supernova explosion. One explosion scenario of a type Ia supernova starts with a deflagration stage: a flame bubble originates near the center of a white dwarf, then buoyantly rises to the surface of the star. While the properties of the flame are known [1], the initial size of the bubble and its location within the star remains uncertain.

In whole-star simulations (see for example [2, 3]) the focus is placed on the late-time development of the bubble, the integral characteristics of the explosion, and the search for a mechanism for transition to detonation. The initial size and location of the bubble are specified somewhat arbitrarily since the true conditions are unknown. Typically, the bubble is assumed to be small, spherical, and at rest. The initial radius is partly determined by the affordable mesh resolution. The placement of the bubble varies from the center of the star, as in [2], to one-tenth of the star radius in [4]. Yet the initial conditions can no longer be assumed irrelevant — it was shown recently that even slightly off-centered ignition produces a highly asymmetric flame bubble followed by asymmetric explosion [3].

In the whole-star simulations to date the flame thickness is unresolvable on the scale of the star, so front tracking or flame capturing techniques are used to propagate the flame with a specified speed. The flame speed, although known for laminar flame, is
altered with various turbulent subgrid flame models. Because of the coarse resolution, and consequently the large size of the initial bubble, the gravitational forces and fluid properties might vary significantly on the scale of the bubble. Considering the complexity of multi-physics whole star simulations is it unreasonable to rely on such simulations in explaining the behavior of small individual bubbles. On the other hand, a stand-alone study of isolated flame bubbles can provide a whole-star simulation with better choice of initial conditions.

In this paper we study an isolated flame bubble rising in a constant gravitational field. We use a simplified model and abstract interpretation of the problem intended for general analysis. The physical properties of a particular system are described in Section 8 where we apply our results to a nuclear flame bubble rising inside a white dwarf star.

The first part of our model is the Boussinesq approximation for buoyancy. In terrestrial flame applications, this approximation is rarely used because of the large density jump between burned and unburned gases in hydro-carbon flames. However, the density difference across the astrophysical flame of interest is small (about 10%) which makes the Boussinesq model suitable. Obviously, the Boussinesq model cannot capture all physical features of the system, such as the Landau-Darrieus instability, thermal expansion, and sound waves. However, it does isolate the interaction between buoyant convection and flame propagation [5, 6], and it does so in a clean and simple enough way to be mathematically tractable [7].

In addition to the Boussinesq approximation we use a model for reaction. In Section 8 we will show that near the center of the star the gravity is weak, so the flame retains its laminar structure and laminar speed (thin flame regime). In the thin flame regime, the flame can be modelled by flame tracking or capturing methods, or by a simplified reaction mechanism. We take the latter approach: the burning at the surface of the bubble is modelled as an one-step reaction.

Both the thin flame assumption and the Boussinesq assumption could be relaxed. The work on buoyant flames by Bell and Zingale [8, 9, 10] is also inspired by the supernova problem. Using a low Mach number code, they performed direct numerical simulations with a detailed nuclear reaction network to study Landau-Darrieus and Rayleigh-Taylor instabilities in astrophysical flames. In the Froude number regime they consider, the gravity was strong enough to create a flow capable of distorting the internal flame structure, i.e. conditions which exist far from the center of the star. We hope that in the future we can compare our work with fully-resolved, low Mach number simulations of flame bubbles.

The paper is organized as follows. We describe the model in the next section and the numerical setup and simulation parameters in Section 3. In Section 4 we discuss the initial evolution of flame bubbles and, using simple dimensional analysis, we estimate the critical size of the bubbles when they deviate from spherical shape. In Section 5 we analyze the late evolution of the bubbles and obtain the conditions for laminar flame propagation (the thin front regime). In Section 6 we focus on the transition from the spherical shape to the thin front solution; we conclude that in a specific range of parameters all bubbles develop according to the same scenario and focus on this scenario in more detail. Section 7 is devoted to the comparison between reacting and non-reacting bubbles and to the comparison with literature. We apply our results to astrophysical bubbles in Section 8 and summarize in Section 9.
2. The governing equations

For combustion, the Boussinesq buoyancy model corresponds to the limit of an infinitely small density difference between burned and unburned states. The fluid velocity obeys the incompressible Navier-Stokes equation with a temperature-dependent force term. The evolution of the temperature, \( 0 \leq T \leq 1 \), is described by an advection-reaction-diffusion equation, which is coupled to the fluid motion through the advection velocity. The governing equations are:

\[
\bar{\rho} \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla p + \nu \nabla^2 \mathbf{v} + f(T),
\]

\[
\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa \nabla^2 T + \frac{1}{\tau} R(T),
\]

\[
\nabla \cdot \mathbf{v} = 0.
\]

Here, \( \bar{\rho} \) is the (constant) average density, \( \nu \) is the kinematic viscosity, \( \kappa \) is the thermal diffusivity, \( \tau \) is the reaction timescale, and \( R(T) \) is the reaction rate. In chemical combustion, Eq. (2) is often used to describe a model system with \( Le = 1 \) and two species, reactant and product. Then, \( T \) can be interpreted both as the temperature and the mass fraction of the burned material [11]. Here, we use the temperature as a reaction progress variable, whose purpose is to distinguish burned, unburned, and partially burned states, and to provide a simple mechanism for flame propagation. Forces depending on the state can be introduced to the model as functions of temperature.

Since flow is assumed to be incompressible, the fluid crossing the reaction front does not undergo thermal expansion and the reaction does not directly affect the velocity. A planar reaction front propagating with constant speed in a motionless fluid is a valid solution in a Boussinesq system without external forces. We refer to this solution as a laminar flame. The properties of the laminar flame are determined by the thermal diffusivity \( \kappa \), reaction time scale \( \tau \) and reaction rate \( R(T) \). The laminar flame has finite thickness, of the order of \( \delta = \sqrt{\kappa \tau} \) and propagates with the speed of the order of \( s = \sqrt{\kappa / \tau} \). The temperature distribution inside the flame front depends on the functional form of reaction rate \( R(T) \).

Unless otherwise specified, we use the Kolmogorov-Petrovskii-Piskunov (KPP) reaction rate [12, 14],

\[
R(T) = \frac{1}{4} T(1 - T).
\]

The reaction rate (3) is well studied mathematically as a source term in reaction-diffusion systems [15]. The KPP reaction rate is capable of generating fronts with speed \( s \geq \sqrt{\kappa / \tau} \); nevertheless, in systems with compact support (localized flames) the laminar front propagates with the minimal speed \( s = \sqrt{\kappa / \tau} \). Although the front is relatively wide (approximately 18\( \delta \) if measured as a distance between two level sets \( T = 0.1 \) and \( T = 0.9 \)), the KPP reaction handles interactions with advecting flow well.†

† Reaction rates of ignition type, for example, can be quenched when the temperature redistributed by the flow drops below ignition threshold. Unlike ignition, the KPP flame cannot be quenched. Depending on the phenomenon being modelled, quenching might be a desirable property, however we are interested in the regime of thin front propagation, when the flame moves with laminar speed with respect to the flow. For thin front modelling, the distribution of temperature inside the front, and therefore the choice of reaction rate, is not important, as long as the flame speed is maintained. This makes KPP a reasonable choice for our reaction rate.
In the coupled system, the flame is affected by the flow through the advection term in the temperature equation, and affects the flow through the force term in Navier-Stokes equation. In our case, the force is gravity:

\[ f(T) = \bar{\rho}g(1 - 2AT), \]

where \( A \equiv \Delta \rho/2\bar{\rho} \ll 1 \) is the Atwood number, and \( g \) is the gravitational acceleration, which is assumed to be vertical and pointing in negative direction, \( g = (0, -g) \).

We consider two-dimensional (2-D) bubbles (three-dimensional cylinders) and three-dimensional (3-D) bubbles with axial symmetry. In two dimensions, Eqs. (1)-(2) with (4) can be expressed in the vorticity-streamfunction formulation, \( \omega = \nabla \times \mathbf{v} = -\nabla^2 \psi \),

\[
\frac{\partial \omega}{\partial t} + (\mathbf{v} \cdot \nabla) \omega = \nu \nabla^2 \omega + 2Ag \frac{\partial T}{\partial x},
\]

\[
\frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla) T = \kappa \nabla^2 T + \frac{1}{\tau} R(T).
\]

Note that in a Boussinesq system the Atwood number and the gravitational acceleration appear only in the algebraic combination \( Ag \).

In cylindrical coordinates \((r, \theta, z)\), the Navier-Stokes and temperature equations for axisymmetric flow can be expressed in terms of angular vorticity, \( \omega = \nabla \times \mathbf{v} \), as follows,

\[
\frac{\partial \omega_r}{\partial t} + v_r \frac{\partial \omega_r}{\partial r} + v_z \frac{\partial \omega_r}{\partial z} - v_r \frac{\omega}{r} = \\
\nu \left[ \frac{\partial^2 \omega_r}{\partial r^2} + \frac{\partial^2 \omega_r}{\partial z^2} + \frac{1}{r} \frac{\partial \omega_r}{\partial r} - \frac{\omega}{r^2} \right] + 2Ag \frac{\partial T}{\partial r},
\]

\[
\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + v_z \frac{\partial T}{\partial z} = \\
\kappa \left[ \frac{\partial^2 T}{\partial r^2} + \frac{\partial^2 T}{\partial z^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right] + \frac{1}{\tau} R(T).
\]

Here the velocities are given by the streamfunction,

\[ v_r = -\frac{1}{r} \frac{\partial \psi}{\partial z}, \quad v_z = \frac{1}{r} \frac{\partial \psi}{\partial r}. \]

The incompressibility constraint, \( \nabla \cdot \mathbf{v} = 0 \), reduces to the elliptic equation which connects the vorticity and the streamfunction,

\[ \omega = -\frac{1}{r} \left[ \frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} \right]. \]

3. Numerical setup

We solve Eqs. (5)-(6) and Eqs. (7)-(8) numerically. The solution is advanced in time as follows: a third order Adams-Bashforth integration in time advances \( \omega \) and \( T \), where spatial derivatives of \( \omega \) and \( T \) are approximated by fourth-order (explicit) finite differences. The subsequent elliptic equation for \( \psi \) is then solved by the bi-conjugate gradient method with stabilization, using the AZTEC library [16]. Finally, we take derivatives of \( \psi \) to update \( \mathbf{v} \). The spatial resolution is \( \Delta x = \Delta y = \delta \) and \( \Delta r = \Delta z = \delta \), which is sufficient to fully resolve the flame in the conditions of interest here. The timestep is set by advective and diffusive CFL limits.
The temperature is initially zero everywhere except for a small spherical (circular in 2-D) spot of burned fluid. The interface between burned and unburned fluid is thickened to match the laminar flame width. The initial velocity field is quiescent. In most cases, taking advantage of symmetry, we solve only half of the domain. Nevertheless, we performed a number of 2-D Cartesian simulations in the full domain with symmetric initial conditions. The solutions were essentially identical to the solutions obtained in a half of the domain. The boundary conditions are reflection (free-slip) in the horizontal direction, and reflection or periodic in the vertical direction. In large enough domains, the choice of top and bottom boundary conditions only marginally affect the solution; we slightly prefer the periodic boundary conditions because of the reduced number of control parameters (initial location of the bubble).

We minimize the effect of initial conditions by starting with a very small flame bubble (with initial radii \( r_0 = 2\delta \) to \( r_0 = 8\delta \)), and we minimize the effect of boundaries by choosing computational domain as large as possible (up to \( 2048 \times 8192 \delta \) for the computational domain containing half of the bubble). Still, as the bubble grows, the influence of boundaries becomes a concern, especially for high viscosities. We verify our results by running the same numerical experiments in domains with different sizes.

Since we are resolving the reaction front, the reaction time \( \tau \) and reaction length scale \( \delta \) are the most natural units of simulation time and space. The rest of the simulation parameters can be represented by two non-dimensional quantities: the Prandtl number and the non-dimensional gravity,

\[
Pr = \frac{\nu}{\kappa}, \quad G = \frac{2Ag \delta}{s^2}.
\]

The non-dimensional gravity \( G \) can be also interpreted as the inverse square of Froude number \( Fr = s/\sqrt{2Ag} \).

One of the purposes of this work is to monitor the influence of the parameters \( G \) and \( Pr \) on the solution, especially for the small \( G \) and small \( Pr \) limits, the regime relevant to the astrophysical flame bubbles.

### 4. Initial evolution of the flame bubble

In this section we use dimensional analysis to estimate the conditions when a small, initially spherical bubble becomes distorted by a constant gravitational field. In the absence of gravity the bubble grows radially with the laminar flame speed. If gravity is non-zero but small, at some time, \( t_{cr} \), the bubble reaches critical size, \( r_{cr} = st_{cr} \), and loses its symmetry.

In a fluid with high viscosity, \( Pr \gg 1 \), the critical size of the bubble is determined by the equilibrium between viscous and gravitational forces. Comparing the gravitational timescale \( \tau_{grav} = \sqrt{r/Ag} \) with the viscous timescale \( \tau_{visc} = r^2/\nu \) we obtain the following scaling,

\[
r_{cr} \sim \left( \frac{\nu^2}{2Ag} \right)^{\frac{1}{2}} = (Pr G)^{\frac{1}{2}} \delta \frac{G}{Pr} \equiv r^*, \quad \text{for} \quad Pr \gg 1.
\]

When the viscosity is low, \( Pr \ll 1 \), the equilibrium between viscous and gravitational forces is never achieved. Indeed, the bubble accelerates until it reaches the terminal velocity which depends on the size of the bubble and increases as the bubble grows, as time squared: \( v_{max} \sim \frac{Ag}{\nu} r^2 \sim \frac{Ag}{\nu}(st)^2 \). The actual velocity of the bubble, however, increases linearly with time, \( v \sim Ag t \), and remains lower than the terminal velocity.
Figure 1. Flame bubbles rising in gravitational fields with $G = 1/4$ (top), $G = 1$ (middle), and $G = 4$ (bottom). Temperature distribution and velocity streamlines are shown. The initial radius of the bubble is $r_0 = 5$; the Prandtl number is $Pr = 1$. The time is shown in units of $\tau$. The size of the computational domain is $256 \times 512 \delta$; the whole domain is shown.
Still, the flame bubble in low viscosity fluids might undergo the stage of radial growth. If gravity is low, it takes time for the bubble to accelerate. Gravitational effects become noticeable when the vertical displacement of the bubble becomes comparable to the size of the bubble, $Agt^2 \sim r_{cr} \sim st$, so that

$$r_{cr} \sim \frac{s^2}{2Ag} = \frac{\delta}{G} \equiv r^*, \quad \text{for} \quad Pr \ll 1. \quad (10)$$

The coefficients in the proportionality relations (9) and (10) must be determined numerically. Motivated by an astrophysical application, we are mostly interested in the low viscosity case. Following the growth of the bubbles in simulations with low Prandtl numbers and different gravities, we estimate the critical radius of the bubble, $r_{cr} \approx 10r^*$ for both 2-D and 3-D bubbles.

To obtain the critical radius, we implicitly assumed that the gravity is small enough so that the evolution of the bubble starts with radial growth. Now we can reformulate this requirement in more qualitative terms. The bubble goes through the stage of radial growth if the critical size of the bubble is larger than the laminar flame thickness, $r^* \gtrsim \delta$. This requirement sets the restrictions for gravity: $G \lesssim Pr^2$ for $Pr \gg 1$, and $G \lesssim 1$ for $Pr \ll 1$. Two inequalities can be combined in the single condition§,

$$G \lesssim \max(1, Pr^2). \quad (11)$$

In Figure 1 we compare cases which do and do not satisfy this condition, for $Pr = 1$. The initial radial growth of the bubble is observed for $G = 1/4$ but not for $G = 4$.

If condition (11) is satisfied, all bubbles with initial radii smaller than critical grow radially until they reach the critical size. The further evolution of the bubbles depends on the critical size $r_{cr}$, but not on the initial size $r_0$. This is illustrated in Fig. 2, where we show the evolution of the bubbles with different initial radii, $r_0 = 2, 4,$ and $8$, at different gravities, $G = 1/4, 1/2,$ and $1$. The solutions for different $r_0$ are plotted with solid, dashed, and dotted lines, and are almost indistinguishable. Also, notice that the larger the gravity $G$, the smaller the size of the bubble at the moment when it becomes distorted.

The statement that the evolution of the bubble at $t > t_{cr}$ does not depend on initial conditions can be extended to small non-spherical bubbles. When $v = 0$, the front propagates normal to itself and consumes the concave sections of the interface. This effect is enhanced by the dependence of the local flame speed on flame curvature $K$, $s_{eff} = s(1 - \mu K)$, where $\mu \sim \delta$ is the Markstein length. As a result, imperfections in initial conditions are smoothed, and if the initial bubble was much smaller than critical, by the time it reaches the critical size, the shape of the bubble is almost spherical.

5. The thin front limit

The thin front limit or the thin flame limit is the limit when the thickness of the flame front is small compared to the size of a typical front feature: in our case, the size of

§ Taking into account the uncertainty of dimensional analysis, we omit coefficients of the order of unity in (11); otherwise Eq. (11) would be $G \lesssim \max \left( \ell, \frac{\ell^2}{C^2} Pr^2 \right)$, where $\ell$ is the width of the laminar flame in units of $\delta$ and $C$ and $c$ are the proportionality coefficients in $r_{cr} \sim r^*$ for high and low Prandtl numbers respectively. As mentioned in the text, in our low Prandtl number simulations we observed $c \approx 10$; and flame thickness depends on reaction rate (for RPP $\ell \approx 18$).
Figure 2. Bubble evolution in laminar flame units. Position and shape of the 2-D bubble (above) and the 3-D bubble (below) at times $t = 16\tau, 32\tau, 48\tau$ for three different gravities (from left to right: $G = 1/4, 1/2, 1$). In each frame, bubbles of three initial sizes are shown: $r_0 = 2\delta$ (solid line), $r_0 = 4\delta$ (dashed line), and $r_0 = 8\delta$ (dotted line). For comparison purposes, the time $t = 0$ is adjusted so that at $t/\tau = 16$ the bubble has horizontal width $2r/\delta = 32$. The border of the bubble is defined as the level set $T = 0.5$. The size of the full computational domain (containing half of the bubble) is $256 \times 512\delta$. 
**Figure 3.** Bubble evolution in thin front units. Position and shape of the 2-D bubble (above) and the 3-D bubble (below) at times $t = Gt/\tau = 16, 32, 48$ for three different gravities (from left to right: $G = 1/4, 1/2, 1$). The border of the bubble is defined as the level set $T = 0.5$. The size of the full computational domain (containing half of the bubble) is $1024 \times 2048 \delta$. 

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the bubble. In the thin front limit, the system behavior is determined by fluid flow on the large scales. The flame is advected by the flow, and propagates with the laminar speed with respect to the flow.

In the thin flame limit, the flame thickness parameter vanishes from all relations. For instance, in the single mode Rayleigh-Taylor configuration discussed in [5], the speed and the shape of the front in the thin flame regime depend on the laminar flame speed, gravitational acceleration and the wavelength of instability, but not on flame thickness $\delta$.

Once $\delta \to 0$ is assumed, there is no natural length scale in the problem, and furthermore, no natural time scale. Indeed, reaction time $\tau = \delta/s$ vanishes with $\delta$; and diffusivity $\kappa = s\delta$ becomes infinitely small to compensate for the fast reaction rate, resulting in the infinitely small viscosity, $\nu = \kappa \text{Pr}$. The only dimensional parameters available for constructing time and space units are gravity $g$ and the flame speed $s$, and there is only one way to do so without involving $\delta$,

$$\tilde{t} = t \frac{2Ag}{s}, \quad \tilde{x} = \frac{x 2Ag}{s^2}.$$  

We refer to units $s^2/2Ag$ and $s/2Ag$ as thin front units, as opposed to the laminar flame units $\delta$ and $\tau$. The relationships between the laminar flame units and thin front units are:

$$\tilde{t} = G \frac{t}{\tau}, \quad \tilde{x} = G \frac{x}{\delta}.$$  

The comparison between solutions expressed in the flame units and in the thin front units is illustrated in Fig. 2 and Fig. 3. Three flame bubbles, rising at gravities $G = 1/4, 1/2, 1$, are shown in both figures. In Figure 2, we plot the flame interface at time intervals $16\tau$ and scale the distances in units of $\delta$, so that Fig. 2 represents the evolution of the bubble in laminar flame coordinates. In Figure 3 we plot data from the same simulations, but now we choose equal time intervals in $\tilde{t} = Gt/\tau$, and scale the distances by $\delta/G$. Thus, Fig. 3 represents the evolution of the bubbles in thin front coordinates. Clearly, there is more similarity among the second set of images.

To better understand the similarity between rescaled solutions, we recast Eqs. (5)-(6) in the thin front units (this can be done with Eqs. (1)-(2) or Eqs. (7)-(8) in a similar way),

$$\frac{\partial \tilde{\omega}}{\partial \tilde{t}} + (\tilde{\mathbf{v}} \cdot \tilde{\nabla}) \tilde{\omega} = \text{Pr} G \tilde{\nabla}^2 \tilde{\omega} - \frac{\partial T}{\partial \tilde{x}},$$  

$$\frac{\partial T}{\partial \tilde{t}} + (\tilde{\mathbf{v}} \cdot \tilde{\nabla}) T = G \tilde{\nabla}^2 T + \frac{1}{G} R(T).$$  

When $G \ll 1$, the right hand side of the temperature equation (13) describes the thin front propagation [17], and can be replaced by $-\delta \nabla T = -\nabla T$; the temperature equation becomes independent of $G$. If, in addition, $\text{Pr} \lesssim 1$, the dissipation term in equation (12) is negligible and the vorticity equation does not depend on $G$. By neglecting the dissipation term in (12) we also lose the dependence of Prandtl number, so the system (12)-(13) becomes parameter-free, with the solution completely determined by the initial state. Combining conditions $G \ll 1$ and $\text{Pr} G \ll 1$ we obtain the criterion for the thin flame propagation

$$G \ll \min(1, \text{Pr}^{-1}).$$  

If (14) is satisfied, any solution rescaled in thin front coordinates depends only on initial conditions and not on parameters $G$, $\text{Pr}$, or the functional form of $R(T)$. In the rest of this section we illustrate this statement using numerical simulations.
5.1. Independence from non-dimensional gravity

The independence of rescaled solutions from the non-dimensional gravity $G$ was mentioned earlier in this section and was illustrated in Fig. 3. The interfaces of the bubbles are alike but not identical — the values of $G = 1, 1/2, 1/4$ used in these simulations are not small enough. Still, the shapes of these bubbles and that for $G = 1/8$, all overlaid in Fig. 4, show the trend to convergence.

5.2. Independence from Prandtl number

We have concluded that in the thin flame regime the solutions with different $Pr \lesssim 1$ are indistinguishable from each other and from solutions with $Pr = 0$. In Figure 4 we compare two bubbles, with $Pr = 1/4$ and with $Pr = 1$, rising in gravity $G = 1/4$. Again, the value of $G$ is not small enough for the solution to be truly in the thin front.
regime; consequently, the solution is affected by the Prandtl number, but weakly.

5.3. Independence from reaction term

Rescaled in the thin front coordinates, the solutions for different $G \ll 1$ have similar shape but differ in the width of the interface. The independence of the interface shape from the width can also be interpreted as independence of the amplitude of the reaction rate $\tau^{-1}$, as long as the reaction rate is balanced by diffusion. In general, we expect the solution to be independent of the functional form of the reaction rate: if $G$ is small, the velocity changes on the scale of the flame thickness are small, and convection transports the flame without affecting the flame structure.

To confirm this, we have attempted to perform a simulation of a rising flame bubble using a different reaction rate. Namely, we use the ignition-type reaction rate, $R(T) = 2(1 - T)$ on the temperature interval $0.5 < T < 1$ and zero otherwise. This reaction rate produces a flame which propagates with the same speed as the KPP flame, but the front is approximately one-fourth as thick. Equivalently resolved
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ignition-type flames therefore require quadrupled resolution and make the simulation up to 256 times more expensive. At quadruple resolution and with $G = 1/4$ we observed the same solution as in the KPP case up to $t \leq 16$. Unfortunately, we could not afford to continue this simulation, and a simulation at reduced resolution produced a bubble with different behavior.

The ignition reaction rate turns out to be a very insidious reaction rate. On the one hand, the thin flame visually suggests the solution is truly in the thin flame regime. On the other hand, this solution is very sensitive to mesh resolution, and it is difficult to converge. The top-left image in Fig. 5 shows our typical calculation with the KPP reaction rate and $\Delta x = \delta$, and below it, the same calculation with the ignition reaction rate. The ignition case looks more accurate since the thin front is visually unaffected by the flow. However if we repeat the ignition-rate simulation at four times higher resolution (the low-right image) we observe a bubble with a different shape. On the other hand, reducing resolution in the KPP simulation (top-right image) produces essentially the same result as in the resolved case. Since the two simulations with the ignition reaction rate differ, no argument for convergence can be made and neither simulation can be trusted. Note that $G = 1$ in these simulations which is too high for the thin flame regime, and independence of solutions from the form of the reaction rate is not expected.

6. The similarity solution

In Section 4 we considered small bubbles, with the size of the order of flame thickness. We have shown that if the gravity is low enough to satisfy condition (11), the evolution of such bubbles starts with radial growth.

In Section 5 we considered the evolution of the bubbles at the thin flame limit, e.g. the bubbles much larger than the flame thickness. We concluded that the thin flame regime requires low gravity, specified by inequality (14). Since condition (11) is weaker than condition (14), in the thin flame regime all initially small bubbles undergo the stage of radial growth (Fig. 6).

In this section, we study the transition between the radial growth stage and the thin flame stage. We focus on the case of $Pr \ll 1$. The transition occurs at time $t_{cr} = r_{cr}/s$, when the bubble reaches critical size $r_{cr}$. The critical size of the bubble (10) and the critical time for the low viscosity and low gravity case can be expressed in the thin front units as

$$\tilde{r}_{cr} = G \frac{r_{cr}}{\delta} = \frac{r_{cr}}{r^*} \sim 1, \quad \tilde{t}_{cr} = G \frac{t_{cr}}{\tau} = \frac{t_{cr}}{t^*} \sim 1. \quad (15)$$

In physical units, the critical radius (10) and the critical time are expressed as follows,

$$r_{cr} \sim r^* = \frac{s^2}{2Ag}, \quad t_{cr} \sim t^* = \frac{s}{2Ag} \quad (16)$$

At the time the bubble reaches the critical size, the solution is already in the thin flame regime. Indeed, from (15) we see that, for $G \ll 1$, the critical size is much larger than the laminar flame thickness, $r_{cr} \gg \delta$. According to (16), the critical time and the critical radius do not depend on flame thickness, the parameter which does not exist in the thin flame limit. Note that for low viscosity the critical time and the critical size of the bubble (15) are invariant in the thin front units; this does not hold for high viscosity fluids (recall (9)).
Figure 6. Regimes of bubble evolution. The gray shaded region corresponds to the radial growth regime (11), i.e. to the regime where an initially small bubble undergoes the stage of linear growth. The region with the striped pattern corresponds to the thin flame regime (14), i.e. to the regime where locally the front preserves the laminar structure. Note that the striped area is completely confined within the gray area. A small initial bubble originating in the thin flame regime, after the stage of radial growth, evolves according to thin flame laws. Small bubbles originating inside the square with dashed outline are characterized by critical time and critical size which are invariant in thin front coordinates (15); such bubbles obey the similarity solution.

The state of the system at the end of the first stage can be considered as initial conditions for the second stage. The thin flame stage always starts from the same initial conditions — the spherical bubble with radius $\tilde{r}_{cr}$ (Eq. 15). The further evolution of the bubble obeys the thin front law, namely propagation with laminar speed with respect to the flow. Since the thin front solution is completely determined by initial conditions (we have discussed this in Section 5), and since initial conditions are the same for all bubbles, all bubbles continue to develop according the same thin front solution.

The similarity between all initially small bubbles at $G \ll 1$ and $Pr \ll 1$, is the most important result of this paper. We will refer to the thin front solution describing all such bubbles as the similarity solution.

Note that “small initial bubbles” does not necessarily mean the radius is of the order of the flame thickness. To develop according to the similarity solution, all that is required is that the initial radius be less than the critical radius (which can be quite large in laminar flame units). A bubble with initial size larger than critical still develops according to the thin front laws, but its shape and velocity distribution might be affected by the initial conditions. Such a solution is still a thin flame solution, but not the similarity solution.
Figure 7. The interface of the 2-D bubble (left) and the 3-D bubble (right) at times $\tilde{t} = Gt/\tau = t/t^* = 48, 72, 96, 120$. For both, $G = 1/4$ and $Pr = 1$, and the size of the full computational domain (containing half of the bubble) is $2048 \times 8192 \delta$ ($512 \times 2048$ thin front units).
Long-term asymptotics for the similarity solution

To obtain long-term asymptotics for the similarity solution, we set up larger simulations of a 2-D bubble and a 3-D bubble. The parameters are the same in both cases: $G = 1/4$, $Pr = 1$, and KPP reaction rate. We chose the size of computational domain containing half of the bubble to be $512 \times 2048$ thin front units ($2048 \times 8192$). Basically, these simulations are larger versions of the two simulations discussed in the previous sections — compare the flame interface shown in Fig. 3, first column, with the results of the longer run shown in Fig. 7 and Fig. 8.

The larger computational domain allowed us to achieve the longer times needed to find the growth rate of the bubble. We measure the height of the apex of the bubble with respect to the original position of the bubble, $h$, and the radius of the bubble (or half-width in 2-D) at the widest point, $r$. Both lengths are shown in Fig. 9 as functions of time. Even though the 2-D and 3-D bubbles have different shapes, the radius and

**Figure 8.** The temperature distribution and velocity streamlines for the bubbles shown in Fig. 7. The snapshots are taken at times $t = 122$ for the 2-D bubble (left) and at $t = 120$ for the 3-D bubble (right).
the height scale as a power law of time with the dimensionality-independent exponent,

\[ \tilde{h} = a t^{2}, \quad \tilde{r} = b t^{9/8}. \]

The coefficients are measured as \( a = 0.086 \) and \( b = 0.655 \) for the 2-D bubble, and \( a = 0.117 \) and \( b = 0.463 \) for the 3-D bubble.

The scaling for the vertical size of the bubble qualitatively agrees with the scaling for the Rayleigh-Taylor instability, \( h = \alpha A t^2 \), where \( \alpha = 2a \). However, the value of Rayleigh-Taylor growth parameter \( \alpha \) for the flame bubbles is an order of magnitude higher: \( \alpha = 0.17 \) for the 2-D bubble and \( \alpha = 0.23 \) for the 3-D bubble. The difference in bubble shape between 2-D and 3-D bubbles — thinner tail and wider and more pronounced vortex cap for the 2-D bubble compared to elongated shape of the 3-D bubble — is also consistent with the topology of the Rayleigh-Taylor instability [18].

The cap of the bubble expands sideways with the speed of the order of laminar flame speed. Expansion in the tail is slower — the reason for this is negative radial (horizontal in 2-D) velocity at the surface of the bubble in the tail region (Fig. 8). Interestingly, the vertical velocity at the lowest point of the bubble does not exceed laminar flame speed, so that the location of initial seed bubble remains inside the bubble boundary at later times, or in other words, the bubble “stays in place”.

7. Discussion

The purpose of this section is to connect our work to the research of others, and thereby to improve our understanding. We discuss experiments and numerical simulations as
verification of our work. We mention theoretical results, even if they are not directly applicable, to stimulate the theoretically-inclined reader; fortunately, a number of models and techniques have been developed in past studies of rising bubbles, falling drops, and gravitational effects in reactive flows.

A large fraction of the existing work is devoted to steady configurations and the search for stabilizing mechanisms. In two-fluid systems, such as air bubbles rising in water and water drops falling through air, the viscosity and the surface tension play a stabilizing role as they prevent bubbles and drops from breaking apart [19]. For the steadily rising flame bubbles [20, 21] the important stabilizing factors are cooling and quenching, which prevent flame bubbles from growing in size. Understanding a stable localized flame is difficult even without considering gravity. Such localized flames, called flame balls [22], were experimentally observed [23] and theoretically understood [24, 25, 26] in the early 90s. Later in the 90s the effect of gravity on a flame ball was studied both experimentally [27] and theoretically [28, 29]. The drift velocity of the flame ball in low gravity was determined, $v \propto \sqrt{gR}$, and found in agreement with the terminal velocity of non-burning bubbles.

The only records we found on non-steady buoyant flame bubbles refer us to the flame strings, which are related to flame balls. The flame strings [27, 30] are long, nearly cylindrical flame structures, observed in experiments on flame balls in microgravity conditions conducted on aircraft. As it appears in videorecordings, the "g-jitters" can stretch a flame ball into a flame-string: one side of the flame-ball is fixed, while the other is pulled from the ball, similar to what we see in our simulations, Fig. 7. The string is slender and its diameter varies with time or axial distance, on a lengthscale of approximately one string diameter. Buckmaster [30] suggested the model which predicts the wavelength of instability. According to this model, the shortest possible wavelength is approximately 5 string diameters, and the fastest growing wavelength is approximately 8 diameters. When the model accounts for the flow divergence, it predicts shorter wavelengths more consistent with the experiment.

The flame string experiment was conducted close to the flammability limit, and the theory was developed accordingly. The threshold in the reaction rate, and consequently the possibility of extinction, are essential for the flame ball and flame string models. In models with the KPP reaction rate, steady flame balls are impossible since any localized initial data will eventually spread out and grow. Even for the ignition rate we use, the radius of the stable flame ball [22] is of the order several $\delta$ and is much smaller than the flame bubbles considered here. Similarly, we cannot expect the exact correspondence between flame strings and our results, but encouraged by the similarity between the two systems, we hope the Buckmaster model could be adapted to our problem.

Besides flames, buoyant reactive flows have been studied in liquid systems with autocatalytic reactions. Typically such systems have a small density difference between the product and reactant fluids so that the Boussinesq approximation is valid. Zhu [31] and Rogers and Morris [32] use autocatalytic reactions to study non-steady reactive bubbles and plumes in the Boussinesq limit. Both the experimental work [32] and numerical study [31] motivated by an experiment are conducted in the high viscosity regime ($Pr \gg 1$), but differ in the Froude number. In [31] the burning dominates over gravitational effects ($G \ll 1$). During the simulation the bubble rises a fraction of its diameter and exhibits the Rayleigh-Taylor instability at the top surface. In [32] gravity prevails over burning ($G \gg 1$), which results in a sequence of long-stem plumes with flat caps. The top surfaces of the caps show no sign of the Rayleigh-
Taylor instability. Such different behavior of two systems so close to ours emphasizes
the sensitivity to the physical parameters and warns us that the extension of our
results beyond their applicability limits must be done with great caution.

Our simulations are done at much lower viscosities, \( \text{Pr} \sim 1 \), and cannot be directly
compared to [31] and [32]. It is not clear whether the similarity solution would develop
Rayleigh-Taylor fingers at the top of the bubble if we extended the simulations to later
times; it is possible that the fingers would be smeared out by shear. In Fig. 7, the
cap of 2-D bubble is no longer convex at the upper surface; this might be considered
as a sign of developing instability. However, we observed similar behavior in the
earlier simulations (Fig. 1, the bottom row), where the solution was influenced by the
boundaries of computational domain. When the simulation was repeated in a larger
domain, the bubble retained its more aerodynamic shape.

The literature is relatively sparse on reacting bubbles but quite rich on non-
reacting bubbles, so in the rest of this section we take a closer look at the latter.
We omit steadily rising bubbles since they are governed by a different stabilization
mechanism, and focus on non-steady bubbles suddenly released from rest. Such
bubbles, at the initial stages of their accelerated motion, provide us with a benchmark
against which we can verify our simulations. Indeed, for flame bubbles much larger
than the critical size, the relative increase in the bubble radius is small, so the influence
of burning is negligible. It can be eliminated completely by solving the equations
without the reaction term. In addition, by turning reaction on and off, we can study
the influence of burning on the on the bubble behavior.

7.1. Initial rise of non-reacting bubbles: literature overview

The simplest possible model, which provides the correct initial acceleration of the
bubble, is the model of a non-deformable bubble rising in ideal flow. The model
accounts only for transformation of potential energy into the kinetic energy of
irrotational fluid inside and outside the bubble. The energy of circulating motion
inside the bubble is neglected, so the model could be applied to liquid and gas bubbles
and drops, and to rigid particles. The energy of displaced fluid is proportional to its
mass, with a coefficient which depends only on the shape of the particle (1/2 for the
sphere and 1 for the circular cylinder, see the chapter on added mass in [33]). If \( m \)
is the ratio of fluid densities inside and outside the drop (or bubble), the drop moves
with initial acceleration,

\[
\begin{align*}
\text{sphere: } \quad a_0 &= -g \frac{m - 1}{m + \frac{1}{2}}, \\
\text{cylinder: } \quad a_0 &= -g \frac{m - 1}{m + 1}.
\end{align*}
\]

(17)

For a heavy drop falling through gas (\( m \gg 1 \)) the initial acceleration is \(-g\); for
the gas bubble rising in liquid (\( m \ll 1 \)) the initial accelerations are \( 2g \) and \( g \)
for the spherical and the cylindrical bubbles, respectively; for the two fluids with close
densities (\( m = 1 - 2A \)) the initial accelerations are,

\[
\begin{align*}
\text{sphere: } \quad a_0 &= \frac{2}{3} (2Ag), \\
\text{cylinder: } \quad a_0 &= \frac{1}{2} (2Ag).
\end{align*}
\]

(18)

The analytical solution for an accelerating viscous spherical drop was derived by
Chisnell [34] (see also review [35]), who used the small Reynolds number assumption
to ensure that the drop remains spherical even in the absence of surface tension.
According to Chisnell, at time \( t = 0 \), acceleration reduces to (17) and the flow pattern
matches the ideal flow around sphere, independent of the density and viscosity of the
inner and outer fluids. Similar to the added mass model, the Chisnell model does not account for deformation of the drop or the bubble.

The initial deformation of accelerating bubbles was studied by Walters and Davidson [36, 37], both theoretically and experimentally. In the experiment, cylindrical and spherical bubbles of air were instantaneously released in water and photographed as they rose. The theory employs the model of a massless incompressible bubble with no surface tension. Both theory and experiment suggested an initial acceleration of $g$ for cylindrical bubbles and $2g$ for the spherical bubbles, and a kidney-shaped cross-section of the bubble.

With the development of numerical methods, rising bubbles and falling drops became benchmark problems for interface tracking methods. For example, Han and Tryggvason, in their study of the later stages of drop development and break-up [38], simulate the initial stages of an instantly released spherical drop with small density difference. This is the regime where Boussinesq approximation is valid, which makes the results of [38] suitable for comparison to our flame bubbles.

### 7.2. Initial rise of reacting and non-reacting bubbles: our simulations

In this subsection we compare two kinds of numerical simulations: with the reaction turned on (as usual) and with the reaction term turned off. Turning off the reaction is equivalent to setting flame speed to zero, so, to avoid ambiguity of flame units, in this section we scale time and distance with gravity and initial size of the bubble, $\hat{t} = t \sqrt{2Ag/r_0}$ and $\hat{x} = x/r_0$. The third dimensional parameter, diffusivity, becomes $\hat{\kappa} = \kappa (2Agr_0^3)^{-\frac{1}{2}}$ in new units. To minimize the influence of reaction and diffusion, we consider bubbles with large initial diameter. Numerical solutions at times $\hat{t} = 1, 2, 3,$ and $4$ are shown in Fig. 10 for cylindrical and spherical bubbles. The simulations are performed at $\hat{\kappa} = \frac{1}{512}$ which, for the burning bubble, corresponds to $G = 1$ and $r_0 = 64\delta$ in flame units.

First, to verify our simulations, we compare the behavior of the non-reacting bubble with the results from the literature discussed above. The overall shape of the bubbles at a given time agrees with the simulations of Han and Tryggvason [38] in the Boussinesq limit. The initial deviation from spherical and cylindrical shapes also agrees with the theory of Walters and Davidson [36, 37] as long as the theoretical solution exists (up to $\hat{t} = 1$ and $\hat{t} = 0.5$ for the cylindrical and the spherical bubbles, respectively). From the location of center of mass (Fig. 11) we determine the initial acceleration $a_0/(2Ag)$ and find it lower than expected: 0.43 instead of 1/2 for the 2-D bubble (cylindrical bubble) and 0.58 instead of 2/3 for the 3-D bubble (spherical bubble). This cannot be explained by viscous effects: according to Chisnell [34], the effect of viscosity should be negligible for the parameter values chosen (Fig. 11). The reasons for slower initial acceleration are the diffused interface, leading to the lower effective Atwood number, and the relatively small computational domain. When we repeat our simulation starting with a larger initial bubble and in a larger computation domain, we obtain initial accelerations which agree with the theory.

Next, we compare the reacting bubbles with non-reacting bubbles. When reaction is turned on, the volume of the bubble increases with time; the top of the bubble, however, rises only marginally faster. The newly burned material accumulates in the cap of the bubble and behind the bubble, in the form of the tail. The tail behind the 3-D bubble is wider than the tail behind the 2-D bubble; below we offer a possible explanation.
Figure 10. Comparison of the bubbles with and without the reaction term switched on and off in 2-D (left) and in 3-D (right). The snapshots are at times $t = 1, 2, 3, 4$ (top to bottom). For the reacting bubble, the times correspond to $t/\tau = 8, 16, 24, 32$; the initial size of the bubble is $r_0 = 64$, the gravity is $G = 1$, the Prandtl number is $Pr = 1$, and the size of computational half-domain is $256 \times 512$. The whole domain is shown.

The flow outside the bubble is ideal (incompressible and irrotational). Vorticity is generated at the flame interface and, for the spherical bubble, inside the bubble, and transported at the local velocity $v$. At the flame front, vorticity transport is slower than the local flame speed, $v + sn$. As a result, all vorticity is confined within the bubble and is zero everywhere outside the bubble.

At the early stages, the flow outside the spherical bubble is the same as the ideal flow around sphere [34]. The surface of the sphere separates internal and external fluid, so that the earlier burned material remains inside the sphere. Newly burned material outside the sphere is swept by the flow to the wake of the bubble where it accumulates. The volume of burned fluid accumulated per unit time is proportional to the flame
Model flames in the Boussinesq limit: Rising bubbles

8. Astrophysical application of flame bubbles: Type Ia supernova ignition

In this section we apply our results to small flame bubbles ignited in a white dwarf at different distances from the center, $Z$. The purpose of this exercise is to estimate the size of the bubble when it loses its spherical symmetry and the time when it happens.

We use the following data, collected in the Table 1. From the cold white dwarf model [3] we take the gravitational acceleration and the density as functions of $Z$. From detailed nuclear deflagration simulations [1] we interpolate the flame parameters — the speed, the thickness, and the Atwood number at relevant densities. Based on this information we compute the estimates of the critical size \( r_{cr} \sim r^* = s^2/2Ag \) and the critical time \( t_{cr} \sim t^* = s/2Ag \), from Eq. (10).

First, since our results apply only in the thin flame regime, we demonstrate
that inside the white dwarf the thin flame conditions (14) are satisfied. Near the center of the star, the dimensionless gravity, \( G = 2Ag/s^2 \), increases proportionally to the distance from the center as \( G \approx 10^{-11}Z/1\text{km} \). Further from the center the density drops, the flame thickness increases and the flame speed decreases, however the resulting \( G \) does not exceed \( 10^{-5} \). The Prandtl number is of the order of \( Pr = 10^{-7} \). Thus, \( G \ll \min(1, Pr^{-1}) \) — everywhere inside the star the flame is well within the thin flame regime.

Next, we take a closer look at our estimates for \( r^* \) and \( t^* \). That would tell us how long a growing bubble would retain its spherical shape and what size it would reach if conditions were uniform and undisturbed. As shown in Fig. 4, the bubble loses its symmetry when it reaches the size \( r_{cr} \approx 10\,r^* \) at time \( t_{cr} \approx 10\,t^* \). Near the center of the star \( r^* \approx 100\,(Z/1\text{km})^{-1}\text{km} \) and \( t^* \approx (Z/1\text{km})^{-1}\text{s} \). Thus, bubbles originating within a few kilometers of the center would grow up to several hundred kilometers in radius and remain spherical. The critical size becomes smaller with distance: the bubbles originating at 10 km, 100 km, and 500 km from the center would lose their spherical symmetry reaching 70 km, 6 km, and 0.3 km in radius respectively.

We have assumed above that the bubble grows in a uniform environment, i.e. in the fluid with constant density and at constant gravitational acceleration. Initially, when the bubble size is still of the order of the flame thickness, this assumption is valid. However, it fails sooner or later, when the bubble reaches the size comparable to the distance from the center of the star. The further from the center, the larger the gravity, the smaller the thin front units are, and the longer (in thin front units) the bubble growth resembles the similarity solution. Consider the bubble which originates close to the center. By the time it starts rising, the bubble is already too

<table>
<thead>
<tr>
<th>( Z ) (km)</th>
<th>( g ) (km/s^2)</th>
<th>( \rho ) (g/cm^3)</th>
<th>( s ) (km/s)</th>
<th>( \delta ) (cm)</th>
<th>( 2A )</th>
<th>( t^* ) (s)</th>
<th>( r^* ) (km)</th>
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</thead>
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<tr>
<td>10</td>
<td>( 5.64 \times 10^3 )</td>
<td>( 2.00 \times 10^9 )</td>
<td>75</td>
<td>( 7.5 \times 10^{-5} )</td>
<td>0.15</td>
<td>0.088</td>
<td>6.64</td>
</tr>
<tr>
<td>20</td>
<td>( 1.12 \times 10^4 )</td>
<td>( 2.00 \times 10^9 )</td>
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<td>( 7.5 \times 10^{-5} )</td>
<td>0.15</td>
<td>0.045</td>
<td>3.35</td>
</tr>
<tr>
<td>50</td>
<td>( 2.80 \times 10^4 )</td>
<td>( 1.98 \times 10^9 )</td>
<td>74</td>
<td>( 7.6 \times 10^{-5} )</td>
<td>0.15</td>
<td>0.017</td>
<td>1.30</td>
</tr>
<tr>
<td>100</td>
<td>( 5.50 \times 10^4 )</td>
<td>( 1.93 \times 10^9 )</td>
<td>72</td>
<td>( 8.1 \times 10^{-5} )</td>
<td>0.15</td>
<td>0.009</td>
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</tr>
<tr>
<td>200</td>
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<td>( 1.75 \times 10^9 )</td>
<td>65</td>
<td>( 9.8 \times 10^{-5} )</td>
<td>0.16</td>
<td>0.004</td>
<td>0.25</td>
</tr>
<tr>
<td>500</td>
<td>( 1.77 \times 10^5 )</td>
<td>( 0.92 \times 10^9 )</td>
<td>32</td>
<td>( 3.5 \times 10^{-4} )</td>
<td>0.20</td>
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</tr>
<tr>
<td>1000</td>
<td>( 1.37 \times 10^5 )</td>
<td>( 0.16 \times 10^9 )</td>
<td>5</td>
<td>( 1.2 \times 10^{-2} )</td>
<td>0.32</td>
<td>0.010</td>
<td>0.09</td>
</tr>
</tbody>
</table>
large and its shape is already affected by the non-uniform gravity and by pressure and density stratification. Still, using the results of this study we can comment on issues related to the whole-star simulations.

In a number of whole-star simulations [2, 3], the flame is artificially thickened by the flame capturing model. So, both the simulated flame thickness and the initial size of the bubble are much larger than the physical flame thickness. We conclude that this should not affect the solution. Since the flame is in the thin flame regime, the solution does not depend on the flame thickness. Since the bubble grows radially at the beginning, choosing a different initial size would result only in a time offset. The last statement requires that a bubble below the critical size is used for the initial conditions, a requirement which is easy to satisfy taking into account how large the critical size is at the relevant gravities.

The viscosity is negligible in astrophysical conditions, so solving the Euler equations is appropriate. However, numerical viscosity is unavoidable in inviscid simulations and is often a concern. We have shown that (a) the astrophysical flame at white dwarf conditions is in the thin flame regime; and (b) in the thin flame regime, the inviscid solution does not differ from solutions with small viscosities, e.g. with $\Pr \equiv v/\kappa \lesssim 1$. In the flame capturing models, the diffusivity is artificially increased and always exceeds the numerical viscosity. We do not expect an inviscid simulation based on such a flame capturing model to be affected by numerical viscosity.

9. Conclusion

A rising flame bubble can be rescaled to the similarity solution, if (i) the initial size of the bubble is small enough, (ii) the initial flow is quiescent, (iii) the Prandtl number is small, and (iv) the non-dimensional gravity is small.

When these conditions are satisfied, the evolution of the bubble can be divided into two stages. The bubble grows radially in an essentially motionless fluid in the first stage. During the second stage, the bubble rises and is distorted by the flow; however the distortion does not affect the internal structure of the flame, and the flame propagates in the thin flame regime. The transition between the two stages occurs when the bubble reaches the critical size $r_{cr} \sim r^* = s^2/(2Ag)$, at time $t_{cr} \sim t^* = r^*/s = s/(2Ag)$, after which the bubble evolves into a mushroom-like structure.

In the thin front units, the critical size is invariant: the evolution of the bubble during the second stage does not depend on the history of the bubble during the first stage or on the parameters ($G$ and $Pr$). That is, the solution is a similarity solution. A physical solution can be rescaled to the similarity solution by using $r^*$ and $t^*$ as units of length and time.

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