Abstract

In 1976, Stephen Hawking proposed that black holes could act as "sinks of information," since it could be possible that a hole would evaporate into a nearly thermal bath of radiation which would not depend on the details of the formation of the hole. Such a proposal seemed to imply that the fundamental laws of quantum mechanics needed to be rewritten, since the time evolution of quantum mechanical states would not be unitary in general; i.e. pure states would evolve into mixed states. In this work, we evaluate how much information is lost, or equivalently, how much entropy is generated when a quantum-mechanical "object" is tossed into a hole much smaller than the spatial extent of the object. We find, in accord with Hawking's arguments, that there is always a (very small) increase of entropy and thus loss of information.

More precisely, we consider the effect of non-trivial scattering on the final purity of an initially pure bosonic quantum state which is scattered off a black hole. For simplicity, we consider a single mode $\phi_{\omega lm}$ of an incoming scalar field on a Schwarzschild geometry, prepared at past null infinity in a single particle state (angular eigenstate $Y_{lm}$, and energy eigenstate $\hbar \omega$), in the limit that the transmission coefficient $\Gamma_{\omega lm} \ll 1$. The final entropy is of first order in $\hbar \omega / \tau_{bh}$, where $\tau_{bh}$ is the Hawking temperature of the black hole, demonstrating that even for low-energy modes, we cannot eliminate the loss of purity.
1. Introduction

Shortly following his discovery of black hole radiance in 1974, Hawking produced a thought experiment in which the unitarity of quantum mechanics seemed to be violated [1]. This long-standing problem, which has come to be known as the black hole information paradox, has attracted a large amount of discussion over the past couple of years, due in large part to the high-powered technology of two-dimensional black hole physics pioneered by Strominger, Giddings, Callan, and Harvey, among others [2].

Since the suggestion of a non-unitarity super-S matrix seemed distasteful to many researchers, others have tried to wriggle out of the paradox in different ways, as summarized quite succinctly by Preskill [3]. The conservative approach to the problem, advocated most notably by ’t Hooft [4], insists that physical mechanisms exist which allow one to recover the information contained in an initial state. Recently, working on this same track, in a mechanism-independent manner, Bekenstein has argued that the information content of the initial state could be contained in the outgoing radiation without violating the laws of thermodynamics [5]. Schier has taken this argument one step further and has produced results which demonstrate that the information paradox is not a problem for low energy bosonic modes due to the amplification of the initial state by stimulated emission processes [6]. In order to simplify his analysis, he has neglected scattering processes in his calculation. In this paper, we will generalize his work to the consideration of both stimulated emission and scattering processes.

We discuss how scattering processes can encode information about an initial state impinging upon a black hole. In the membrane paradigm advanced by Thorne et al. [7], we can understand this process in a very simple, intuitive manner. An initial quantum state prepared at past null infinity is visualized as being a quantum mechanical wavepacket which propagates on a 3+1 spacetime, endowed with a metric defined on the 3-space $g_{ij}$ and a time lapse function $\alpha(r,t)$ which measures the tick rate of a clock of a fiducial observer (FIDO) at a given point compared to that of an observer at infinity. The presence of the black hole corresponds to the presence of a scattering potential with transmission coefficient $s_{lml}$, where $s$ is the spin of the particle species, $\hbar \omega$ is the energy of the particle at spatial infinity, $l$ is the orbital quantum number, and $m$ is the azimuthal quantum number. This transmission coefficient then affects both the outgoing radiation emitted from the membrane and the incoming quantum state. When $s_{lml} = 1$, the radiation is exactly thermal, since particles incident on the hole are completely absorbed while all particles radiated from the hole escape to infinity. When $s_{lml} = 0$, particles radiated by the hole are completely confined within the atmosphere of the hole, whereas all incident particles travel on to infinity unaffected by the barrier. We expect that in the limit $s_{lml} \rightarrow 0$, we will be able to recover the purity of the final quantum state.

2. Calculation of the Final Entropy

We consider an incoming state populated with a single particle of spin $s$ in a mode with energy $\hbar \omega$, orbital angular eigenvalue $l$, and azimuthal eigenvalue $m$. Because the transmission coefficient $s_{lml}$ has a fast drop-off with increasing $l$, the chance of a particle penetrating the hole’s potential barrier will be greatest for an $l = 0$ state. Once a particle has been absorbed by the black hole, the details of its initial configuration will be lost to an observer on the outside of the hole. Thus, consideration of the $l = 0$ mode, with its maximum likelihood of particle absorption, will lead to a greater amount of information loss than any other value of $l$. We could also consider a case in which more than one particle occupies the initial state. When such an initial state scatters from the hole, the problem becomes more difficult to solve, since the presence of many particles complicates the combinatorics of the scattering process. By considering only a single particle initial state, we are able to describe accurately the essential physical process without overcomplicating the analysis. We define the quantity of entropy $H_{lml}^{\omega}$ in the outgoing state in terms of the occupation probabilities of the outgoing state $p_{lml}(n)$, where $n$ represents the number of particles in a given mode with the specified energy and angular eigenvalues.

$$H_{lml}^{\omega} = -\sum_{n=0}^{\infty} p_{lml}(n) \log[p_{lml}(n)]$$  \hspace{1cm} (1)

If we imagine an observer who records only the occurrence probabilities of the outgoing states in an ensemble of identical black hole systems, eq. (1) represents the entropy of the final state in the ensemble, as calculated...
by such an observer. The fact that this definition also physically corresponds to the entropy induced in a
density matrix by tracing over the degrees of freedom on the future horizon \( H^+ \) is evident from inspection of
a Penrose diagram of a Schwarzschild black hole, since we compute the entropy over all outgoing final states,
which is equivalent to computing the entropy induced from a density matrix traced over the incoming states
on the horizon. These outgoing probabilities are connected to the incoming state through a conditional
probability factor, given by

\[
p_o^{\ell m}(n') = \sum_{n=0}^{\infty} p_0^{\ell m}(n'|n)p_i^{\ell m}(n).
\]

This expression carries the incoming probabilities to the outgoing probabilities, playing a role analogous
to that of the S matrix scattering amplitude expression \(<\phi_{\text{out}}|S|\phi_{\text{in}}>\) in quantum field theory. The matrix
\( p_i^{\ell m}(n'|n) \) can also be interpreted in the context of communication theory as a source of noise. We can see
this fact because if \( p_i^{\ell m}(n'|n) \) does not map every \( p_i^{\ell m}(n) \) back onto a unique \( p_i^{\ell m}(n) \), the system mixes up
initial probabilities in a way that cannot be disentangled. The conditional probability appearing in Eq. (2) has been decomposed into a piece representing the contribution from scattering processes and
another piece representing spontaneous and stimulated emission [8].

\[
p^{\ell m}(n'|n) = \sum_{k=0}^{\min(n',n)} p_{\text{scat}}^{\ell m}(k|n)p_{\text{spont}+\text{stim}}^{\ell m}(n' - k|n)
\]

More precisely : \( p_{\text{scat}}^{\ell m}(k|n) \) is the probability that, when \( n \) photons come in, \( k \) of them get scattered
off the potential barrier into the hole's thermal atmosphere; and \( p_{\text{spont}+\text{stim}}^{\ell m}(j|n) \) is the probability that
these same \( n \) incoming photons and the accompanying vacuum fluctuations of their incoming \( \ell m \) mode
will interact with the hole's thermal atmosphere to produce \( j \) outgoing photons via stimulated and sponta-
aneous emission. (The vacuum fluctuations trigger the spontaneous emission; the \( n \) incoming photons trigger
the stimulated emission.) It is remarkable (as Bekenstein and Meisels showed) that the probabilities for absorption/scattering [embodied in \( p_{\text{scat}}^{\ell m}(k|n) \)] are independent of the probabilities for triggering emission [embodied in \( p_{\text{spont}+\text{stim}}^{\ell m}(j|n) \)]; their independence is embodied in the simple product that appears in Eq.
(3). Note that in Eq. (3), the summation up to \( \min(n',n) \) assures that no more particles will be scattered
than are present in the initial state. However, due the the presence of stimulated and spontaneous emission,
there is always some probability that the total number of particles in the final state is greater than that
present in the initial state. Our choice of the initial state requires that

\[
p_i^{\ell m}(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \neq 1. \end{cases}
\]

Combining eqs. (2-4), we get

\[
p_o^{\ell m}(n) = \sum_{k=0}^{1} p_{\text{scat}}^{\ell m}(k|1)p_{\text{spont}+\text{stim}}^{\ell m}(n - k|1).
\]

Physically, eq. (5) tells us that there are two mechanisms which produce \( n \) particles in the outgoing
radiation in a mode in which one particle is incident. The \( k = 0 \) term represents the initial particle being
absorbed and \( n \) particles being created through spontaneous or stimulated emission, while the \( k = 1 \) term
represents the reflection of the initial particle with \( n - 1 \) particles emitted.

The forms of \( p_{\text{scat}}^{\ell m}(k|n) \) and \( p_{\text{spont}+\text{stim}}^{\ell m}(k|n) \) have been determined separately by both field-theoretic
and information-theoretic methods [8] :

\[
p_{\text{scat}}^{\ell m}(k|n) = \binom{n}{k} \Gamma^{n-k}(1 - \Gamma)^k,
\]

\[
p_{\text{spont}+\text{stim}}^{\ell m}(k|n) = \binom{k+n}{n} (1 - e^{-\gamma})^{n+1} e^{-\gamma^k}.
\]
Here $\gamma$ is defined through the relation
\begin{equation}
\frac{1}{e^{\gamma} - 1} = \frac{\Gamma}{e^x - 1},
\end{equation}
where $x = \hbar\omega/\tau_{bh}$, with $\tau_{bh}$ as the Hawking temperature of the black hole. Next, we make the approximation that the absorption of particles by the black hole is very small, $\Gamma \ll 1$. This approximation is valid when $\hbar\omega/\tau_{bh} \ll 1$. Intuitively, this corresponds to the limit when the wavepacket is highly non-localized, so that only a small portion of the incoming state actually feels the presence of the black hole. We can expand $\Gamma$ in a power series of $x$, where for low-energy modes, the leading term is of order $x^2$ —
\begin{equation}
\Gamma_{x=0} \simeq \frac{(s!)^4x^2}{4\pi^2} = Ax^2.
\end{equation}
Substituting Eq. (9) into Eq. (8), we obtain
\begin{equation}
\gamma = \log\frac{e^x - 1}{Ax^2} + 1 \simeq \log\frac{1}{Ax} + 1.
\end{equation}
Thus, in the following, we use
\begin{equation}
e^{-\gamma} = \frac{Ax}{1 + Ax}.
\end{equation}

The factor $p_{\text{scat}}^{\omega lm}(k|n)$ in eq. (6) takes into account the scattering of the initial state, with each factor of $(1 - \Gamma)$ representing the reflection of a single particle and each factor of $\Gamma$ representing the transmission of a single particle into the hole. The combinatorial factor $\binom{k}{n}$ ensures that $p_{\text{scat}}^{\omega lm}(k|n)$ is properly normalized. The expression $p_{\text{stim} + \text{spon}}^{\omega lm}(k|n)$ in eq. (7) takes into account the spontaneous and stimulated emission of particles, as well as their chance to be transmitted through the black hole potential barrier. Thus, the factor $e^{-\gamma k}$ is a Boltzmann suppression of the emission of $k$ particles, where $\gamma$, defined in Eq. (8), can be thought of as the effective ratio of energy to temperature, including the barrier transmission effect. The combination $\binom{k+n}{n}(1 - e^{-\gamma})^{n+1}$ guarantees that the mean number of quanta emitted is $\frac{(n+1)}{e^{-\gamma}}$, as it should be for the sum of stimulated and spontaneous emission.

Using Eq. (6) and Eq. (7), we determine
\begin{align}
p_{\text{scat}}^{\omega lm}(1|1) &= (1 - \Gamma), 
\end{align}
\begin{align}
p_{\text{scat}}^{\omega lm}(0|1) &= \Gamma, 
\end{align}
\begin{align}
p_{\text{stim} + \text{spon}}^{\omega lm}(k|1) &= (k + 1)(1 - e^{-\gamma})^2e^{-\gamma k}, 
\end{align}
\begin{align}
p_{\text{stim} + \text{spon}}^{\omega lm}(k - 1|1) &= k(1 - e^{-\gamma})^2e^{-\gamma k}e^{\gamma}.
\end{align}

By inserting the approximate relations (9) and (10) into equations (11)-(14), we obtain
\begin{align}
p_{\text{scat}}^{\omega lm}(1|1) &= (1 - Ax^2), 
\end{align}
\begin{align}
p_{\text{scat}}^{\omega lm}(0|1) &= Ax^2, 
\end{align}
\begin{align}
p_{\text{stim} + \text{stim}}^{\omega lm}(k|1) &= (k + 1)(Ax^2)(\frac{Ax}{1 + Ax})^n, 
\end{align}
\begin{align}
p_{\text{stim} + \text{stim}}^{\omega lm}(k - 1|1) &= k(1 - Ax^2)(\frac{Ax}{1 + Ax})^{n-1}.
\end{align}

Inserting these expressions back into Eq. (2), and making use of the approximation $\frac{Ax}{1 + Ax} \simeq Ax(1 - Ax)$, we find the outgoing probability
\begin{equation}
p_{\omega lm}^{\omega lm}(n) = [1 - Ax(1 - Ax)]^2[Ax(1 - Ax)]^{n-1}[n(1 - Ax^2) + (n + 1)Ax^2(Ax)(1 - Ax)].
\end{equation}
Then, using Eq. (1), we calculate the outgoing entropy in terms of the $p_{o}^{\ell m}(n)$ given by Eq. (19). Since $H_{0} = -\sum_{n=0}^{\infty} p_{o}^{\ell m}(n) \log p_{o}^{\ell m}(n)$, and the $p_{o}^{\ell m}(n)$ are a power series expansion in $x$, where $x \ll 1$, we can proceed by writing down the first three terms of the entropy, ultimately keeping only the leading order term in $x$.

\begin{align*}
p_{o}(0) &= [1 - Ax(1 - Ax)] A x^2, \\
p_{o}(0) &\approx [1 - 2 Ax + 3 A^2 x^2] A x^2, \tag{20} \end{align*}

using $[1 - Ax(1 - Ax)] A x^2 \approx 1 - 2 Ax + 3 A^2 x^2 + O(x^3)$. Similarly, we evaluate $p_{o}(1)$ and $p_{o}(2)$.

\begin{align*}
p_{o}(1) &\approx [1 - 2 Ax + 3 A^2 x^2][1 - Ax] + 2(Ax^2)(Ax)(1 - Ax), \\
p_{o}(1) &\approx 1 - 2 Ax + (3 A^2 - A)x^2, \tag{22} \end{align*}

\begin{align*}
p_{o}(2) &\approx [1 - 2 Ax + 3 A^2 x^2][1 - Ax][2(1 - Ax^2) + 3(Ax^2)(Ax)(1 - Ax)], \\
p_{o}(2) &\approx 2 Ax - 6 A^2 x^2. \tag{24} \end{align*}

If we define the $n$th term in the entropy as $B_{n}$, $B_{n} = -p_{o}(n) \log p_{o}(n)$, we need to evaluate $B_{0} + B_{1} + B_{2}$ to leading order in $x$.

\begin{align*}
B_{0} &\approx -Ax^2 + 2 A^2 x^3 \log[Ax^2 - 2A^2 x^3], \tag{26} \\
B_{0} &\approx -Ax^2 (\log[Ax^2] + 2Ax), \tag{27} \\
B_{1} &\approx -[1 - 2 Ax + (3 A^2 - A)x^2] \log[1 - 2 Ax + (3 A^2 - A)x^2], \tag{28} \\
B_{1} &\approx 2 Ax - 4 A^2 x^2, \tag{29} \\
B_{2} &\approx -[2 Ax - 6 A^2 x^2] \log[2 Ax - 6 A^2 x^2], \tag{30} \\
B_{2} &\approx 2 Ax \log[2 Ax] + 6 A^2 x^2 (1 + \log[2 Ax]). \tag{31} 
\end{align*}

Thus, we obtain

\begin{equation}
H_{0} = 2 Ax (1 - \log[2Ax]) + O(x^2). \tag{32} 
\end{equation}

This expression for the entropy contains only the terms $n = 0, 1, 2$. The higher $n$ terms have been left out because for $n > 2$, $p_{o}(n) \approx n(Ax)^{n-1}$ [see eq. (19)], which implies $B_{n} = O(x^{n-1} \log x)$, an expression negligibly small compared to the contribution from $n = 0, 1, 2$. Furthermore, this expression (32) for the entropy of the outgoing state is positive definite in the regime which we are considering, namely $x \ll 1$, since $\log[2Ax] < 0$ for all $x < 1$.

### 3. Discussion

Since we have taken a one particle initial state, our calculation provides an upper bound on the amount of entropy per particle. Intuitively, the presence of many particles in the incoming state will tend to drown out the noise created by the distorted thermal emission of the hole. The generalization of this calculation to that of a many particle initial state is straightforward, but somewhat tedious, and is not necessary for the demonstration of the essential physics occurring here.

Whenever we consider extremely low energy modes, the joint effect of scattering and stimulated emission processes causes the entropy of the final state to drop off significantly faster with decreasing energy than when either process is (unphysically, of course) considered independent of the other. This again agrees with Schiffer's statement that his calculation serves as an upper bound to the actual entropy in the final state. However, since consistency requires that $\Gamma_{n=0} \ll 1$ if $x \ll 1$, we realize that the situation which Schiffer analyzed really amounts to a trivial case (Hawking radiation completely retained in the thermal atmosphere of the hole while incoming states are completely reflected). In addition, once scattering effects are taken into account, the constraint that Schiffer imposes on his incoming beam, $\tilde{n} \gg 1$, can be relaxed, since the spontaneous emission will not penetrate through to an external observer.

Of course, our arguments are valid only in the limit that $x \ll 1$. The black hole information paradox becomes especially strong in the geometrical optics limit, $x \gg 1$, and $\Gamma \rightarrow 1$ which implies that all states are absorbed directly into the hole. In this limit, the black hole seems to be a wastebacket of information.
It seems hard to reconcile this result with Bekenstein’s mechanism-independent argument mentioned in the introduction, since the outgoing radiation remains nearly independent of the details of a high-energy state impinging upon the hole. In the extreme limit of a hole which is formed solely from ultra-high energy states, it would seem that the outgoing radiation should contain no useful information about the incoming state. This point has also been noted by Schiffer, who suggested the existence of currently-unknown physical mechanisms which would allow us to circumvent this difficulty. In regards to this issue, we would like to suggest that part of the difficulty lies in ignoring physical processes which occur when an incoming state nears the horizon, which become significant when the hole is extremely small or the incoming state is very energetic. In a local frame of reference, the physical fields we are familiar with at electroweak scales and below should experience interactions with themselves and with other fields, a mechanism which one totally ignores with the free-field theory approach taken in this paper. If such processes can establish correlations between incoming and outgoing modes, then we may have to include more realistic fields and interactions in order to fully address the loss of purity issue.

Acknowledgements

We would like to thank Eanna Flannagan and Kip Thorne for many useful discussions, and for their help in preparing and reviewing this manuscript.
References