Stochastic Lagrangian Description of Navier-Stokes Equations

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Outline:

1. Language

2. Gautam Iyer’s Thesis

3. Questions
\[ f \in BV[0, T] \iff \sup_{\mathcal{P}} \sum_k |f(t_{k+1}) - f(t_k)| < \infty \]

\[ f \in BV \Rightarrow df \text{ is a Radon measure} \]

\[ \int_0^T \phi df \text{ usual Stieltjes integral} \]

- Brownian motion is not BV.

\[ \mathbb{E} \sum_k |W(t_{k+1}) - W(t_k)| = \infty \]
Stochastic (Itô) Integral

\[
\int_0^t \phi dW = \lim_{N \to \infty} \sum_{k=0}^{N-1} \phi(t_k)(W(t_{k+1}) - W(t_k))
\]

\[
\mathbb{E} \sum_k (W(t_{k+1}) - W(t_k))^2 = T
\]

\(X_t\) continuous \((\mathcal{F}_t)\) adapted. Martingale:

\[E(X_t|\mathcal{F}_s) = X_s, \quad a.s \ t > s.\]

Example:

\[X_t = x + \int_0^t \phi dW\]
Semimartingale:

\[ S_t = M_t + B_t, \text{ (Martingale + BV)}. \]

\[
\int_0^t \phi dS = \lim_{N \to \infty} \sum_{k=0}^{N-1} \phi(t_k)(M(t_{k+1}) - M(t_k)) + \int_0^t \phi dB
\]

Example: SDE

\[ dX = u(X)dt + \sigma(X)dW \]

is the semimartingale eqn

\[ X_t = x_0 + \int_0^t \sigma(X_s) dW + \int_0^t u(X_s) ds \]
Quadratic Variation

\[ \langle X \rangle_t = \lim_{\| \Delta \| \to 0} \sum_{k=0}^{N-1} \left( X_{t \wedge t_{k+1}} - X_{t \wedge t_k} \right)^2 \]

Examples:

- Continuous BV process: \( \langle X \rangle_t = 0 \).
- Brownian motion: \( \langle W \rangle_t = t \).
- If \( X_t \) is a continuous bounded (r. local) martingale then \( \langle X \rangle_t \) exists, and
  \[ X_t^2 - \langle X \rangle_t \]
  is a bounded (r. local) martingale.
Joint Quadratic Variation

\[
\langle X, Y \rangle_t = \lim_{\|\Delta\| \to 0} \sum_{k=0}^{N-1} (X_{t\wedge t_{k+1}} - X_{t\wedge t_k})(Y_{t\wedge t_{k+1}} - Y_{t\wedge t_k})
\]

• If \( M \) is a continuous local martingale and \( f \in L^2(\langle M \rangle) \) then the Itô integral

\[
\int_0^t f dM
\]

is a continuous local martingale and

\[
\langle \int f dM, N \rangle_t = \int_0^t f_s d\langle M, N \rangle_s
\]

\( \forall \ N \) continuous local martingale.
Itô Formula

If $F(x_1, \ldots, x_n, t)$ is deterministic and smooth and $X = (X_1, \ldots X_n)$ is a continuous (vector valued) semimartingale then

$$F(X_t, t) - F(X_0, 0) = \int_0^t \partial_s F(X_s, s) ds + \int_0^t \nabla F(X_s, s) \cdot dX_s + \frac{1}{2} \int_0^t \frac{\partial^2 F(X_s, s)}{\partial x_i \partial x_j} d\langle X_i, X_j \rangle_s$$

In differential form:

$$d(F(X, t)) = (\partial_i F')dX_i + \left( \frac{1}{2} \frac{\partial^2 F}{\partial x_i \partial x_j} d\langle X_i, X_j \rangle + \partial_t F \right) dt$$
Generalized Itô -Wentzell- Bismut- Kunita Formula

• If $F(x, t)$ is a continuous $C^2$ process and a $C^1$ semimartingale, and if $g_t$ is a continuous vector-valued predictable process, then the composition $F(g_t, t)$ is a continuous predictable process and

\[
F(g_t, t) - F(g_0, 0) = \int_0^t \partial_i F(g_s, s) dg^i_s + \frac{1}{2} \int_0^t \partial^2 F(g_s, s) \frac{\partial^2 F(g_s, s)}{\partial x_i \partial x_j} d\langle g^i_s, g^j_s \rangle \\
+ \int_0^t F(g_s, ds) + \langle \int_0^t \partial_i F(g_s, ds), g^i_s \rangle
\]

Here

\[
\int F(g_s, ds) = \lim_{N \to \infty} \sum_{k=0}^{N-1} \left( F(g_{t_k}, t_{k+1}) - F(g_{t_k}, t_k) \right)
\]
\[ \int \partial_i F(g_s, ds) = \lim_{N \to \infty} \sum_{k=0}^{N-1} \left( \partial_i F(g_{t_k}, t_{k+1}) - \partial_i F(g_{t_k}, t_k) \right) \]

are usual Itô integrals.

Proof explains it:

\[ F(g_t, t) - F(g_0, 0) = \sum_{k=0}^{N-1} \left\{ F(g_{t_k}, t_{k+1}) - F(g_{t_k}, t_k) \right\} + \sum_{k=0}^{N-1} \left\{ F(g_{t_{k+1}}, t_{k+1}) - F(g_{t_k}, t_{k+1}) \right\} \]

and

\[ \sum_{k=0}^{N-1} \left\{ F(g_{t_{k+1}}, t_{k+1}) - F(g_{t_k}, t_{k+1}) \right\} = I + J + K \]
with

\[ I = \sum_{k=0}^{N-1} \left\{ \partial_i F(g_{t_k}, t_{k+1}) - \partial_i F(g_{t_k}, t_k) \right\} (g_{t_{k+1}}^i - g_{t_k}^i), \]

\[ J = \sum_{k=0}^{N-1} \partial_i F(g_{t_k}, t_k) (g_{t_{k+1}}^i - g_{t_k}^i) \]

\[ K = \frac{1}{2} \sum_{k=0}^{N-1} \frac{\partial^2 F(\xi_k, t_{k+1})}{\partial x_i \partial x_j} (g_{t_{k+1}}^i - g_{t_k}^i)(g_{t_{k+1}}^j - g_{t_k}^j) \]

\[ \lim_{\|\Delta\| \to 0} I = \langle \int \partial_i F(g, ds), g^i \rangle, \quad \lim_{\|\Delta\| \to 0} J = \int \partial_i F(g, s) dg^i \]

\[ \lim_{\|\Delta\| \to 0} K = \frac{1}{2} \int \frac{\partial^2 F(g, s)}{\partial x_i \partial x_j} d\langle g^i, g^j \rangle. \]
Theorem 1 (C1) Let $W$ be an $n$-dimensional Wiener process. Let $k \geq 1$ and assume $u_0 \in C^{k+1,\alpha}$ is a deterministic divergence-free vector field. Let $(u, X)$ solve the stochastic system

$$
\begin{aligned}
\frac{dX}{dt} &= udX + \sqrt{2\nu}dW, \\
A &= X^{-1}, \\
u &= \text{EP} \left\{ (\nabla^t A) (u_0 \circ A) \right\}
\end{aligned}
$$

Then $u$ solves the deterministic incompressible NSE:

$$
\begin{aligned}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p &= 0, \\
\nabla \cdot u &= 0
\end{aligned}
$$

• When $\nu = 0$ all is deterministic, and we recover the Eulerian-Lagrangian deterministic representation based on the Weber formula.
Remarks

• $A = X^{-1}$ is the spatial inverse ("back-to-labels"). It exists, and it is as smooth as $X$. Both are stochastic.

• Forced NSE

$$ dX = u dt + \sqrt{2\nu} dW, $$

$$ A = X^{-1} $$

$$ u = \mathbb{E} \mathbb{P} \left\{ (\nabla^t A) \left[ u_0 + \int_0^t (\nabla^t X) f(X_s, s) ds \right] \circ A(t) \right\} $$

represents

$$ \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = f, \quad \nabla \cdot u = 0. $$

• Representations for Lans-alpha, Burgers. No direct representation for Leray regularization.
Local Existence for the Stochastic System, Remarkable Formulae

**Theorem 2** Let $u_0 \in C^{k+1,\alpha}$ be divergence-free. There exists a $T > 0$ depending on the norm of $u_0$, but independent of viscosity, so that a solution $(u, X)$ of the stochastic system exists on $[0, T]$. Moreover, $\|u\|_{C^{k+1,\alpha}} \leq U$ for $t \in [0, T]$ with $U$ dependent on the norm of the initial data and $T$.

**Theorem 3** Let $\omega = \nabla \times u$, $\omega_0 = \nabla \times u_0$. Then

$$\omega = \mathbb{E} \{((\nabla X)\omega_0) \circ A\}.$$

In two dimensions,

$$\omega = \mathbb{E} [\omega_0 \circ A].$$
For forced systems in $n = 2, 3$, replace in the formulae above $\omega_0$ by

$$\xi_t = \omega_0 + \int_0^t (\nabla X_s)^{-1} g(X_s, s) \, ds$$

with $g = \nabla \times f$.

- Circulation is conserved.

Let

$$\tilde{u} = \mathbb{P}\{ (\nabla^t A)(u_0 \circ A) \}$$

This is a stochastic incompressible velocity, with initial data $u_0$ and

$$u = \mathbb{E}\tilde{u}$$

$$\oint_{X(\gamma)} \tilde{u} \cdot dr = \oint_{\gamma} u_0 \cdot dr.$$
Stochastic Conservation Laws

• The “back-to-labels” process obeys

\[ dA_t + [u \cdot \nabla A - \nu \Delta A] \, dt + \sqrt{2\nu} \nabla A dW = 0 \]

For any smooth function \( \theta(a, t), \, \nu(x, t) = \theta(A(x, t), t) \) obeys

\[ dv_t + [u \cdot \nabla v - \nu \Delta v] \, dt + \sqrt{2\nu} \nabla v dW = \partial_t \theta \circ A \]

• The proof uses the Itô-Wentzell-Bismut-Kunita formula.

• Cancellation, chain rule as if it were a first order PDE, due to the joint quadratic variation.
Comparison with deterministic Eulerian-Lagrangian Representation

The NSE are equivalent to

\[ \partial_t \overline{A} + u \cdot \nabla \overline{A} - \nu \Delta \overline{A} = 0 \]

\[ u = \mathbb{P}\{ (\nabla^t \overline{A}) v \} \]

\[ (\partial_t + u \cdot \nabla - \nu \Delta) \overline{v}_i = 2\nu C^j_{k;i} \partial_k \overline{v}_j \]

with

\[ C^j_{k;i} = (\nabla \overline{A})^{-1}_{pi} \partial_k \partial_p \overline{A}_j \]

In the stochastic representation

\[ u = \mathbb{E}\mathbb{P}( (\nabla^t A) v ) \]
with \( v = u_0(A) \), obeying the stochastic conservation law

\[
dv_t + [u \cdot \nabla v - \nu \Delta v] \, dt + \sqrt{2\nu} \nabla v dW = 0
\]

Clearly

\[
\overline{A} = \mathbb{E}A
\]

The connection coefficients compensate for the joint quadratic variation.
Idea of Proof of the Main Result

Let

\[ w_i = (\partial_i A) \cdot v \]

Then, by Itô

\[ dw_i = (\partial_i A) \cdot dv + d(\partial_i A) \cdot v + d(\partial_i A_j, v_j) \]

Using the equations of stochastic conservation law for \( v \) and Itô we obtain

\[ dw + \left[ u \cdot \nabla w - \nu \Delta w + (\nabla u)^t w \right] dt + \sqrt{2\nu} \nabla w dW = 0. \]

Integrating and taking expected value finishes the proof.
Questions

- If $X(t, a)$ is a turbulent Lagrangian path observed numerically or experimentally is

$$\langle X \rangle_t = 0?$$

If one considers

$$\langle X^i \rangle_\Delta = \sum_{k=0}^{N-1} (X^i(t_{k+1}) - X^i(t_k))^2$$

how does this depend on $\|\Delta\|$? Does it matter if $X^i$ is replaced with $X \cdot e$ where $e$ is aligned with the mean flow, or perpendicular to it? Does one see the transition to 0 (BV) at $\|\Delta\|$ corresponding to the Kolmogorov time?
• If $a, b$ are two labels, same kind of questions for

\[ \langle X(t, a), X(t, b) \rangle \]

Is it a smooth function of $(a, b)$? How does the transition to 0 depend on the separation between $a$ and $b$?

• Is there a connection between the joint quadratic variation and vortex reconnection? (As suggested by the deterministic Eulerian-Lagrangian calculations of Ohkitani?)
References

- H. Kunita, Stochastic flows and stochastic differential Eqns, CUP(1990)


